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澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試 (語言科及數學科)

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2019 年試題及參考答案
2019 Examination Paper and Suggested Answer

數學正卷 Mathematics Standard Paper

<u>**第一部分**</u> **選擇題。請選出每題之*最佳答案*。**

1. 若集合 $A=\{x : x^2-x-6<0\}$，則 $A=$

A. $\{x:-2<x<3\}$ B. $\{x:x>3$ 或 $x<-2\}$ C. $\{x:-3<x<2\}$
D. $\{x:x>2$ 或 $x<-3\}$ E. $\varnothing$

2. 若 x 隨 $\sqrt{m}$ 正變且隨 n^2 反變，當 m 增加 44% 和 n 減少 20%，x 增加的百分比是多少？

A. 64 B. 92.5 C. 84 D. 68.4 E. 87.5

3. 若 x^3+4x^2+bx+3 能被 x^2+ax+1 整除，則 $a+b=$

A. 1 B. 4 C. 5 D. 7 E. 以上皆非

4. 求 $(6-\sqrt{35})^{100}(6+\sqrt{35})^{99}$ 之值。

A. $99(6+\sqrt{35})$ B. $99(6-\sqrt{35})$ C. $6+\sqrt{35}$
D. $6-\sqrt{35}$ E. 以上皆非

5. 設 $2^a=5$。用 a 來表示 $\log_{10}2$。

A. $\dfrac{1}{a+1}$ B. $a+1$ C. $\dfrac{a}{a+1}$ D. $\dfrac{a+1}{a}$ E. $a-1$

6. 若 $x>2$，則 $\sqrt{x^2-2x+1}+|2-x|=$

A. -1 B. $2x-3$ C. $3-2x$ D. 1 E. $2x-1$

7. 某 10 個連續偶數之和為 430，求當中的最大數。

A. 34 B. 40 C. 46 D. 52 E. 58

8. 直綫 $x+2y+4=0$ 和 $3x-by+8=0$ 相交於 y 軸，求 b 之值。

A. -16 B. -8 C. -4 D. 4 E. 16

9. 設 D 為符合不等式組 $\begin{cases} x\geq 0 \\ y\geq 0 \\ x-y\geq -2 \\ 3x+2y\leq 24 \end{cases}$ 的解所組成的區域。

以下哪些點位於 D 區內 (包括邊界)？

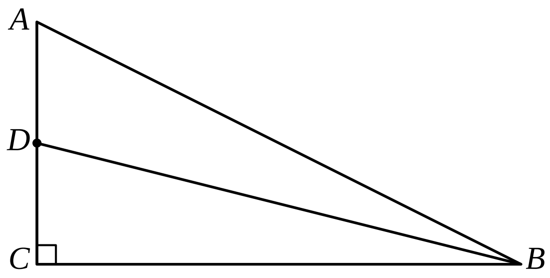
I. (1, 1) II. (4, 6) III. (7, 0)

A. 只有 I 及 II B. 只有 I 及 III C. 只有 II 及 III
D. I、II 及 III E. 以上皆非

10. 用 1、3、5、9 組成的所有無重複數字的四位數的總和是多少？

A. 119988　　B. 17776　　C. 19998　　D. 239976　　E. 319998

11. 圖中 ΔACB 為直角三角形。D 是 AC 的中點，且 $|CB|=2|AC|$。那麼 $\tan\angle ABD=$



A. $\frac{7}{6}$　　B. $\frac{1}{5}$　　C. $\frac{2}{9}$　　D. $\frac{1}{9}$　　E. $\frac{1}{3}$

12. 以方程 $x^2-3x+1=0$ 的兩個根的平方為根的一元二次方程是

A. $x^2-7x-1=0$　　B. $x^2-7x+1=0$　　C. $x^2+7x+1=0$

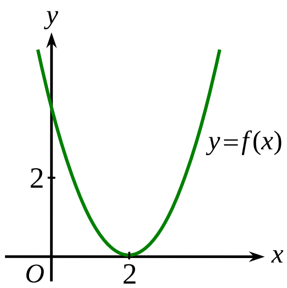
D. $x^2+7x-1=0$　　E. $x^2-x+7=0$

13. 若橢圓 $\frac{x^2}{a+2}+\frac{y^2}{a^2}=1$ 的焦點在 y 軸上，則實數 a 的取值範圍為

A. $(-2,+\infty)$　　B. $(2,+\infty)$　　C. $(-2,0)\cup(1,+\infty)$

D. $(-\infty,-1)\cup(2,+\infty)$　　E. $(-2,-1)\cup(2,+\infty)$

14. 下圖所示為 $y=f(x)$ 的圖像，且圖像的頂點為 $(2,0)$，則以下哪一點是 $y=f(x-3)+1$ 的圖像的頂點?



A. $(-3,0)$　　B. $(-5,0)$　　C. $(3,1)$　　D. $(5,1)$　　E. 以上皆非

15. 設 $P(n)$ 為一道命題，並對所有正整數 n，有 $P(n)\Rightarrow P(n+1)$。若對正整數 m，$P(m)$ 成立，那麼 $P(n)$

A. 對所有正整數 n 都成立　　B. 對所有 $n\geq m$ 都成立　　C. 對所有 $n<m$ 都成立

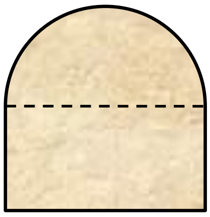
D. 對所有 $n\leq m$ 都成立　　E. 以上皆非

<u>**第二部分**</u> **解答題。**

1. 從 1、2、3、⋯、1000 中，隨意抽出一個正整數。求以下各事件的概率。
   - (a) 被抽出的數的個位是 3 或 7。 (2 分)
   - (b) 被抽出的數不是立方數。 (3 分)
   - (c) 被抽出的數可被 4 或 5 整除。 (3 分)

2. 已知兩點 $A(-1, 2)$ 及 $B(0, 5)$。
   - (a) 若直綫 $y = mx + b$ 為綫段 AB 的垂直平分綫，求 m 和 b。 (4 分)
   - (b) 圓 C 通過 A 和 B 兩點，且圓心在直綫 $2x + y = 5$ 上。求圓 C 的標準方程。 (4 分)

3. 一板身為長方形而頂部則為半圓形 (見下圖) 的石板周界為六米長。問此石板最大可能面積為多少平方米？ (8 分)



4. 等比數列 $\{a_n\}_{n\geq1}$ 的各項均為正數，且 $a_1 + 2a_2 = 1$，$a_4^2 = 4a_3a_7$。
   - (a) 求數列 $\{a_n\}_{n\geq1}$ 的通項公式。 (4 分)
   - (b) 設 $b_n = \log_2 a_1 + \log_2 a_2 + \cdots + \log_2 a_n$，求數列 $\left\{\frac{1}{b_n}\right\}_{n\geq1}$ 的前 n 項和。 (4 分)

5. 已知 $\sin\alpha + \cos\beta = \sqrt{3}$ 及 $\cos\alpha - \sin\beta = 1$。
   - (a) 求 $\sin(\alpha - \beta)$ 的值。 (4 分)
   - (b) 證明 $\cos\left(\frac{\pi}{6} + \beta\right) = 1$。 (4 分)

JM01 數學正卷 – 參考答案

第一部分 選擇題。

題目編號	最佳答案
1	A
2	E
3	C
4	D
5	A
6	B
7	D
8	C
9	D
10	A
11	C
12	B
13	E
14	D
15	B

(第二部分答案由下頁開始)

第二部分 解答題。

1. (a) 從 1 到 1000 之間個位數字為 3 的整數包括 3、13、⋯、993，共有 100 個。同樣地，7、17、⋯、997 也有 100 個個位數字為 7 的整數。

因此 P(個位是 3 或 7)$=\frac{100+100}{1000}=\frac{1}{5}$。

(b) 由 1 到 1000 的立方數包括 $1^3=1$、$2^3=8$、⋯、$10^3=1000$，共10個。

因此 P(不是立方數)$=1-P$(立方數)$=1-\frac{10}{1000}=\frac{99}{100}$。

(c) 共有 $\frac{1000}{4}=250$ 個可被 4 整除的整數和 $\frac{1000}{5}=200$ 個可被 5 整除的整數。此外，共有 $\frac{1000}{4\times 5}=50$ 個可同時被 4 和 5 整除的整數。

因此 P(被 4 或 5 整除)$=P$(被 4 整除)$+P$(被 5 整除)$-P$(被 4 及 5 整除)

$=\frac{250+200-50}{1000}$

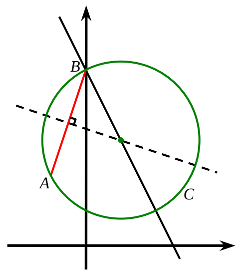
$=\frac{2}{5}$。

2. (a) AB 的中點有座標 $\left(\frac{-1+0}{2},\frac{2+5}{2}\right)=\left(-\frac{1}{2},\frac{7}{2}\right)$。

穿過 $A(-1, 2)$ 和 $B(0, 5)$ 的直綫有斜率 $(5-2)/(0-(-1))=3$。所以 AB 的垂直平分綫有斜率 $-\frac{1}{3}$。

AB 的垂直平分綫的方程為 $\frac{y-7/2}{x-(-1/2)}=-\frac{1}{3}$，即 $y=-\frac{1}{3}\left(x+\frac{1}{2}\right)+\frac{7}{2}=-\frac{1}{3}x+\frac{10}{3}$。

因此有 $m=-\frac{1}{3}$ 和 $b=\frac{10}{3}$。



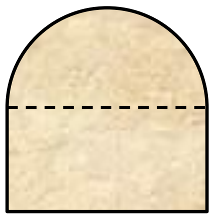
(b) 直綫 $2x+y=5$ 會與 AB 的垂直平分綫(根據 (a)，其方程可寫為 $x+3y=10$)相交於 C 的圓心。圓心的位置可由聯立方程組 $\begin{cases}2x+y=5\\x+3y=10\end{cases}$ 的解求出。

解方程組得圓心的座標為 (1, 3)。

圓形的半徑 r 為 $B(0, 5)$ 到圓心 (1, 3) 的距離，即 $r=\sqrt{(1-0)^2+(3-5)^2}=\sqrt{5}$。

由此知道圓形 C 的標準方程是 $(x-1)^2+(y-3)^2=5$。

3. 設半徑長 r 米，並設長方形高 h 米。由周界得 $\pi r+2h+2r=6$，從而得 $h=3-\frac{\pi+2}{2}r$。



石板面積 $=\frac{\pi}{2}r^2+2rh=\frac{\pi}{2}r^2+2r\left(3-\frac{\pi+2}{2}r\right)$ (米^2)

$=6r-\frac{\pi+4}{2}r^2=\frac{\pi+4}{2}\left(\frac{12}{\pi+4}r-r^2\right)$ (米^2)

$=\frac{\pi+4}{2}\left[\frac{36}{(\pi+4)^2}-\left(\frac{6}{\pi+4}-r\right)^2\right]=\frac{18}{\pi+4}-\frac{\pi+4}{2}\left(\frac{6}{\pi+4}-r\right)^2$ (米^2)

$\le\frac{18}{\pi+4}$ (米^2)。

∴ 最大面積為 $\frac{18}{\pi+4}$ 米^2。

4. (a) 設數列 $\{a_n\}_{n\ge1}$ 的公比為 $q\,(q>0)$。由 $a_4^2=4a_3a_7$ 得 $a_4^2=4a_5^2$，從而得 $q^2=\dfrac{a_5^2}{a_4^2}=\dfrac{1}{4}$，故此 $q=\dfrac{1}{2}$。

由 $a_1+2a_2=1$ 得 $a_1+2qa_1=1$，從而得 $a_1=\dfrac{1}{2}$。因此數列 $\{a_n\}_{n\ge1}$ 的通項公式是 $a_n=a_1q^{n-1}=\dfrac{1}{2^n}=2^{-n}$。

(b) 由 (a) 的結果、對數的定義 ($a_n=2^{-n}\Leftrightarrow\log_2 a_n=-n$) 以及等差級數公式得

$$b_n=\log_2 a_1+\log_2 a_2+\cdots+\log_2 a_n=(-1)+(-2)+\cdots(-n)=-\frac{n(n+1)}{2}\text{。}$$

由此得 $\dfrac{1}{b_n}=-\dfrac{2}{n(n+1)}=-2\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)$，從而數列 $\left\{\dfrac{1}{b_n}\right\}_{n\ge1}$ 的前 n 項和為

$$\begin{aligned}\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_n}&=-2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]\\&=-2\left(1-\frac{1}{n+1}\right)\\&=\frac{-2n}{n+1}\text{。}\end{aligned}$$

5. (a) 由 $\sin\alpha+\cos\beta=\sqrt{3}$，得 $\sin^2\alpha+2\sin\alpha\cos\beta+\cos^2\beta=3$ --------------------------------- (1)

由 $\cos\alpha-\sin\beta=1$，得 $\cos^2\alpha-2\cos\alpha\sin\beta+\sin^2\beta=1$ ------------------------------------ (2)

將 (1)、(2) 相加得 $2+2(\sin\alpha\cos\beta-\cos\alpha\sin\beta)=4$，即 $2\sin(\alpha-\beta)=2$，從而有

$$\sin(\alpha-\beta)=1\text{。}$$

(b) 由 $\sin\alpha+\cos\beta=\sqrt{3}$，得 $\sin^2\alpha=\left(\sqrt{3}-\cos\beta\right)^2=3-2\sqrt{3}\cos\beta+\cos^2\beta$ ------------------ (3)

由 $\cos\alpha-\sin\beta=1$，得 $\cos^2\alpha=\left(1+\sin\beta\right)^2=1+2\sin\beta+\sin^2\beta$ ----------------------------(4)

將 (3)、(4) 相加得 $1=5+2\sin\beta-2\sqrt{3}\cos\beta$，即 $2\sqrt{3}\cos\beta-2\sin\beta=4$。

最後的方程可寫成 $\frac{\sqrt{3}}{2}\cos\beta-\frac{1}{2}\sin\beta=1$，即

$$\cos\left(\frac{\pi}{6}+\beta\right)=1\text{。}$$

<u>Part I</u> Multiple choice questions. Choose the *best answer* for each question.

1. If set $A=\{x: x^2-x-6<0\}$, then $A=$

A. $\{x: -2<x<3\}$ B. $\{x: x>3 \text{ or } x<-2\}$ C. $\{x: -3<x<2\}$
D. $\{x: x>2 \text{ or } x<-3\}$ E. $\varnothing$

2. If x varies directly as $\sqrt{m}$ and inversely as n^2, what is the percentage increase of x when m is increased by 44% and n is decreased by 20%?

A. 64 B. 92.5 C. 84 D. 68.4 E. 87.5

3. If x^3+4x^2+bx+3 is divisible by x^2+ax+1, then $a+b=$

A. 1 B. 4 C. 5 D. 7 E. none of the above

4. Find the value of $(6-\sqrt{35})^{100}(6+\sqrt{35})^{99}$.

A. $99(6+\sqrt{35})$ B. $99(6-\sqrt{35})$ C. $6+\sqrt{35}$
D. $6-\sqrt{35}$ E. none of the above

5. Let $2^a=5$. Express $\log_{10}2$ in terms of a.

A. $\frac{1}{a+1}$ B. $a+1$ C. $\frac{a}{a+1}$ D. $\frac{a+1}{a}$ E. $a-1$

6. If $x>2$, then $\sqrt{x^2-2x+1}+|2-x|=$

A. -1 B. $2x-3$ C. $3-2x$ D. 1 E. $2x-1$

7. The sum of 10 consecutive even numbers is 430. Find the largest number among them.

A. 34 B. 40 C. 46 D. 52 E. 58

8. The lines $x+2y+4=0$ and $3x-by+8=0$ intersect at the y-axis. Find the value of b.

A. -16 B. -8 C. -4 D. 4 E. 16

9. Let D be the region which represents the solution of the system of inequalities: $\begin{cases} x\geq 0 \\ y\geq 0 \\ x-y\geq -2 \\ 3x+2y\leq 24 \end{cases}$.

Which of the following points lie in D (including the boundary)?

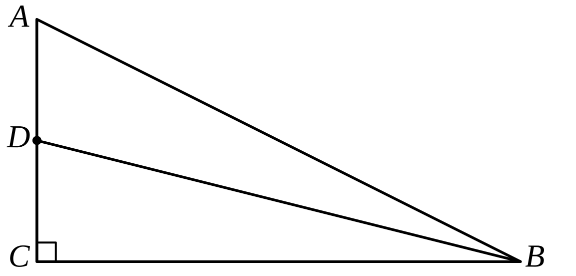
I. (1, 1) II. (4, 6) III. (7, 0)

A. I and II only B. I and III only C. II and III only
D. I, II and III E. none of the above

10. What is the sum of all the 4-digit numbers having digits 1, 3, 5, 9 without repetition?

A. 119988 B. 17776 C. 19998 D. 239976 E. 319998

11. In the figure, ΔACB is a right-angled triangle. D is the midpoint of AC, and $|CB|=2|AC|$. Then $\tan\angle ABD=$



A. $\frac{7}{6}$ B. $\frac{1}{5}$ C. $\frac{2}{9}$ D. $\frac{1}{9}$ E. $\frac{1}{3}$

12. Find a quadratic equation with two roots that are the square of the roots of the equation $x^2-3x+1=0$.

A. $x^2-7x-1=0$ B. $x^2-7x+1=0$ C. $x^2+7x+1=0$

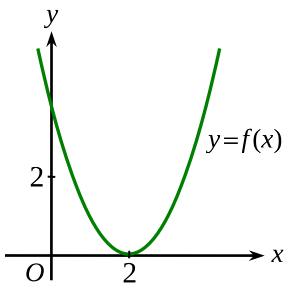
D. $x^2+7x-1=0$ E. $x^2-x+7=0$

13. If the foci of the ellipse $\frac{x^2}{a+2}+\frac{y^2}{a^2}=1$ are on the y-axis, then the range of a is

A. $(-2, +\infty)$ B. $(2, +\infty)$ C. $(-2, 0)\cup(1, +\infty)$

D. $(-\infty, -1)\cup(2, +\infty)$ E. $(-2, -1)\cup(2, +\infty)$

14. The figure below shows the graph of $y=f(x)$, and the vertex of the graph is (2, 0). Which of the following is the vertex of the graph of $y=f(x-3)+1$?



A. (−3, 0) B. (−5, 0) C. (3, 1) D. (5, 1) E. none of the above

15. Let $P(n)$ be a statement such that $P(n)\Rightarrow P(n+1)$ for all positive integers n. If $P(m)$ is true for positive integer m, then $P(n)$ is true for

A. all positive integers n B. all $n\geq m$ C. all $n<m$

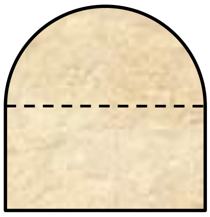
D. all $n\leq m$ E. none of the above

<u>Part II</u> Problem-solving questions.

1. A positive integer is randomly chosen from the numbers 1, 2, 3,…, 1000. Find the probability of each of the following events.
   (a) The chosen integer has unit digit 3 or 7. (2 marks)
   (b) The chosen integer is not a cubic number. (3 marks)
   (c) The chosen integer is divisible by 4 or by 5. (3 marks)

2. Two points $A(-1, 2)$ and $B(0, 5)$ are given.
   (a) Let the line $y = mx + b$ be the perpendicular bisector of the line segment *AB*. Find the values of *m* and *b*. (4 marks)
   (b) A circle *C* passes through the two points *A* and *B*, with its center on the line $2x + y = 5$. Find the standard equation of the circle. (4 marks)

3. A stone tablet of a rectangular body and a semi-disc head (see the figure below) is made with perimeter 6 meters long. What is the largest possible area (in square meters) of the tablet? (8 marks)



4. Given that every term of the geometric sequence $\{a_n\}_{n\geq 1}$ is positive, and that $a_1 + 2a_2 = 1$ and $a_4^2 = 4a_3a_7$.
   (a) Find the general term of $\{a_n\}_{n\geq 1}$. (4 marks)
   (b) Let $b_n = \log_2 a_1 + \log_2 a_2 + \cdots + \log_2 a_n$. Find the sum of the first *n* terms of the sequence $\left\{\frac{1}{b_n}\right\}_{n\geq 1}$. (4 marks)

5. Given that $\sin\alpha + \cos\beta = \sqrt{3}$ and $\cos\alpha - \sin\beta = 1$.
   (a) Find the value of $\sin(\alpha - \beta)$. (4 marks)
   (b) Prove that $\cos\left(\frac{\pi}{6} + \beta\right) = 1$. (4 marks)

JM0l Mathematics Standard Paper – Suggested Answer

Part I Multiple choice questions.

Question Number	Best Answer
1	A
2	E
3	C
4	D
5	A
6	B
7	D
8	C
9	D
10	A
11	C
12	B
13	E
14	D
15	B

(Answers for Part II start from next page)

<u>Part II</u> Problem-solving questions.

1. (a) Among 1 to 1000, integers with unit digit 3 include 3, 13, …, 993, a total of 100 integers. Similarly, among 7, 17, …, 997, there are 100 integers with unit digit 7.

   Thus $P(\text{unit digit is 3 or 7}) = \frac{100+100}{1000} = \frac{1}{5}$.

   (b) Cubic numbers from 1 to 1000 include $1^3 = 1$, $2^3 = 8$, …, $10^3 = 1000$, a total of 10 integers.

   Thus $P(\text{not a cubic number}) = 1 - P(\text{cubic number}) = 1 - \frac{10}{1000} = \frac{99}{100}$.

   (c) There are $\frac{1000}{4} = 250$ integers that are divisible by 4, and $\frac{1000}{5} = 200$ integers that are divisible by 5.

   Also, there are $\frac{1000}{4\times 5} = 50$ integers that are divisible by both 4 and 5.

   Thus $P(\text{divisible by 4 or 5}) = P(\text{divisible by 4}) + P(\text{divisible by 5}) - P(\text{divisible by 4 and 5})$

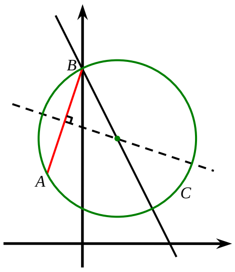
   $$= \frac{250+200-50}{1000}$$

   $$= \frac{2}{5}.$$

2. (a) The mid-point of AB has coordinates $\left(\frac{-1+0}{2}, \frac{2+5}{2}\right) = \left(-\frac{1}{2}, \frac{7}{2}\right)$.

   Slope of the line passing through $A(-1, 2)$ and $B(0, 5)$ is $(5-2)/(0-(-1)) = 3$. So the perpendicular bisector of AB has slope $-\frac{1}{3}$. Hence the equation of the perpendicular bisector of AB is $\frac{y-7/2}{x-(-1/2)} = -\frac{1}{3}$, that is, $y = -\frac{1}{3}\left(x+\frac{1}{2}\right) + \frac{7}{2} = -\frac{1}{3}x + \frac{10}{3}$.

   Thus $m = -\frac{1}{3}$ and $b = \frac{10}{3}$.



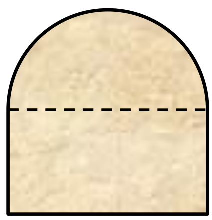
   (b) The line $2x + y = 5$ should meet the perpendicular bisector of AB (from (a), its equation can be written as $x + 3y = 10$) at the center of circle C. The location of the center can be found by solving the system of linear equations $\begin{cases} 2x + y = 5 \\ x + 3y = 10 \end{cases}$.

   Solving this system of linear equations yields the coordinates of the center, namely (1, 3).

   The radius r of the circle equals the distance from $B(0, 5)$ to the center (1, 3). Thus we have $r = \sqrt{(1-0)^2 + (3-5)^2} = \sqrt{5}$.

   It follows from the above that the standard equation of circle C is $(x-1)^2 + (y-3)^2 = 5$.

3. Let the radius be r m long, and let the rectangle be h m high. From the given perimeter, we have $\pi r + 2h + 2r = 6$. Thus $h = 3 - \frac{\pi+2}{2}r$.



   Area of the tablet $= \frac{\pi}{2}r^2 + 2rh = \frac{\pi}{2}r^2 + 2r\left(3 - \frac{\pi+2}{2}r\right)$ (m^2)

   $$= 6r - \frac{\pi+4}{2}r^2 = \frac{\pi+4}{2}\left(\frac{12}{\pi+4}r - r^2\right) \text{ (m}^2\text{)}$$

   $$= \frac{\pi+4}{2}\left[\frac{36}{(\pi+4)^2} - \left(\frac{6}{\pi+4} - r\right)^2\right] = \frac{18}{\pi+4} - \frac{\pi+4}{2}\left(\frac{6}{\pi+4} - r\right)^2 \text{ (m}^2\text{)}$$

   $$\le \frac{18}{\pi+4} \text{ (m}^2\text{)}.$$

   $\therefore$ The largest area of the tablet is $\frac{18}{\pi+4}$ m^2.

4. (a) Let the common ratio of $\{a_n\}_{n\geq 1}$ be q ($q>0$). From $a_4^2 = 4a_3a_7$, we have $a_4^2 = 4a_5^2$. It follows that $q^2 = \frac{a_5^2}{a_4^2} = \frac{1}{4}$, and so $q = \frac{1}{2}$. From $a_1 + 2a_2 = 1$, we have $a_1 + 2qa_1 = 1$, and so $a_1 = \frac{1}{2}$. Hence the general term of $\{a_n\}_{n\geq 1}$ is given by $a_n = a_1q^{n-1} = \frac{1}{2^n} = 2^{-n}$.

(b) From the result of (a), the definition of logarithm ($a_n = 2^{-n} \Leftrightarrow \log_2 a_n = -n$), and the formula for arithmetic series, we get

$$b_n = \log_2 a_1 + \log_2 a_2 + \cdots + \log_2 a_n = (-1) + (-2) + \cdots (-n) = -\frac{n(n+1)}{2}.$$

It follows that $\frac{1}{b_n} = -\frac{2}{n(n+1)} = -2\left(\frac{1}{n} - \frac{1}{n+1}\right)$, and so the sum of the first n terms of $\left\{\frac{1}{b_n}\right\}_{n\geq 1}$ is given by

$$\begin{aligned}\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} &= -2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right] \\ &= -2\left(1 - \frac{1}{n+1}\right) \\ &= \frac{-2n}{n+1}.\end{aligned}$$

5. (a) Since $\sin\alpha + \cos\beta = \sqrt{3}$, we have $\sin^2\alpha + 2\sin\alpha\cos\beta + \cos^2\beta = 3$ ------------------------ (1)

Since $\cos\alpha - \sin\beta = 1$, we have $\cos^2\alpha - 2\cos\alpha\sin\beta + \sin^2\beta = 1$ -------------------------- (2)

Adding up (1) and (2), we get $2 + 2(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = 4$, i.e. $2\sin(\alpha - \beta) = 2$, and so

$$\sin(\alpha - \beta) = 1.$$

(b) Since $\sin\alpha + \cos\beta = \sqrt{3}$, we have $\sin^2\alpha = \left(\sqrt{3} - \cos\beta\right)^2 = 3 - 2\sqrt{3}\cos\beta + \cos^2\beta$ --------- (3)

Since $\cos\alpha - \sin\beta = 1$, we have $\cos^2\alpha = (1 + \sin\beta)^2 = 1 + 2\sin\beta + \sin^2\beta$ -------------------(4)

Adding up (3) and (4), we get $1 = 5 + 2\sin\beta - 2\sqrt{3}\cos\beta$, i.e. $2\sqrt{3}\cos\beta - 2\sin\beta = 4$.

The last equation can be written as $\frac{\sqrt{3}}{2}\cos\beta - \frac{1}{2}\sin\beta = 1$, i.e.

$$\cos\left(\frac{\pi}{6} + \beta\right) = 1.$$